Reg. No. : ....

Code No. : 6371 **Sub. Code : ZMAM 23**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

Second Semester

Mathematics — Core

ADVANCED CALCULUS

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer.

1. Let D be the set of points (x, y) with $0 \le x \le 1$, $0 \le y \le 1$ and both x and y rational. Then

   (a) $\underline{A}(D) = \overline{A}(D) = 0$

   (b) $\underline{A}(D) = 1, \overline{A}(D) = 0$

   (c) $\underline{A}(D) = 0, \overline{A}(D) = 1$

   (d) $\underline{A}(D) = \overline{A}(D) = 1$

2. Let D be the region between the line $y = x$ and the parabola $y = x^2$. Take $f(x, y) = xy^2$. Then $\iint_D f$ is

(a) $\frac{1}{40}$ (b) $\frac{1}{20}$

(c) $\frac{1}{80}$ (d) $\frac{1}{10}$

3. Let $S : \begin{cases} u = x + y \\ v = x - y \\ w = x^2 \end{cases}$ the image of (1, 2) under S is

(a) (3, 1, 1) (b) (1, 1, 0)

(c) (3, –1, 1) (d) (3, –1)

4. Let $T : \begin{cases} u = x^2 + y - z \\ v = xyz^2 \\ w = 2xy - y^2 z \end{cases}$ Then $dT|_{(1,1,1)}$ is

(a) $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & -1 \\ 2 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

5. The Jacobian of the transformation

$T : \begin{cases} u = x \cos y \\ v = x \sin y \end{cases}$ is

(a) $x^2 \sin y$

(b) x

(c) $x \cos y$

(d) x^2

6. The sine and cosine functions are

(a) both linearly dependent and functionally dependent

(b) linearly independent and functionally dependent

(c) linearly dependent but not functionally dependent

(d) linearly independent but not functionally dependent

7. If γ is a smooth curve whose domain is the interval $[a, b]$ then $L(\gamma)$ is given by

(a) $\int_a^b \gamma'(t)\,dt$ (b) $\int_a^b \sqrt{|\gamma'(t)|}\,dt$

(c) $\int_a^b |\gamma'(t)|\,dt$ (d) $\int_a^b |\gamma'(t)|^2\,dt$

8. If γ is a curve of class C'' with arc length as the parameter, then the curvature of γ at the point corresponding to $t = c$ is

(a) $k = |\gamma''(c)|$ (b) $k = |\gamma'(c)|$

(c) $k = \gamma''(c)$ (d) $k = \gamma'(c)$

9. If ω is any differential form of class C'', then ddw is

(a) w (b) 0

(c) $-w$ (d) w^*

10. If $x = u^2 + v$, $y = v$ and $\sigma = xy^2 dxdy$ then σ^* is

(a) $(u^3v^2 + uv^3)dudv$

(b) $2uv^2 dudv$

(c) $(2u^3v^2 + 2uv^3)dudv$

(d) $2uv^3 dudv$

PART B — (5 × 5 = 25 marks)

Answer ALL questions choosing either (a) or (b).

11. (a) Let f and g be continuous and bounded on D prove that $\iint_D (f+g) = \iint_D f + \iint_D g$ and $\iint_D Cf = C\iint_D f$ for any constant C.

Or

(b) Show that for $x > 0$

$$\int_0^{\frac{\pi}{2}} \log[\sin^2\theta + x^2\cos^2\theta]d\theta = \pi \log\left(\frac{x+1}{2}\right)$$

12. (a) Consider the following linear transformations of the plane into itself.

$$S:\begin{cases} u = 2x - 3y \\ v = x + y \end{cases} \quad T:\begin{cases} u = x + y \\ v = 3x + y \end{cases}$$

Find ST and TS and check whether $ST = TS$ or not.

Or

(b) For any $P \in S$ and any $u \in R^3$, prove that $dg|_p(u) = Dg(p) \cdot u$, where g is a real valued function of class C' defined on an open set S in 3-space.

13. (a) Let T be a transformation from R^n into R^n which is of class C' in an open set D and suppose that $J(P) \neq 0$ for each $P \in D$. Prove that T is locally 1-to-1 in D.

Or

(b) Let F and G be of class C' in an open set $D \subset R^5$. Let $p_0 = (x_0, y_0, z_0, u_0, v_0)$ be a point of D at which both of the equations. $F(x, y, z, u, v) = 0$, $G(x, y, z, u, v) = 0$ are satisfied. Suppose also that $O(F, G)/O(u, v) \neq 0$ at p_0. Prove that there are two function ϕ and ψ of class C' in a neighborhood N of (x_0, y_0, z_0) such that $u = \phi(x, y, z)$, $v = \psi(x, y, z)$ is a solution of $F = G = O$ in N giving u_0 and v_0 at (x_0, y_0, z_0).

14. (a) If E is a closed bounded subset of Ω of zero volume, prove that $T(E)$ has zero volume.

Or

(b) Let γ_1 and γ_2 be smoothly equivalent smooth curves, and let p be a simple point on their trace. Prove that γ_1 and γ_2 have the same direction at p.

15. (a) Let $\overline{a} = 2i - 3j + k, \overline{b} = i - j + 3k$, $\overline{c} = i - 2j$. Compute the vectors $(\overline{a} \times \overline{b}) \cdot \overline{c}$ and $a \times (b \times c)$.

Or

(b) If ω is any differential form of class C'', prove that $dd\omega = 0$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If f is continuous on R, prove that $\iint\limits_R f$ exists.

Or

(b) Let R be the rectangle described by $a \le x \le b, c \le y \le d$ and let f be continuous on R. Prove that $\iint\limits_R f = \int_a^b dx \int_c^d f(x, y) dy$.

17. (a) Define a linear transformation. Let L be a linear transformation from R^n into R^m represented by the matrix $[a_{ir}]$. Prove that there is a constant B such that

$|L(P)| \le B|P|$ for all points P.

Or

(b) Let T be differentials on an open set D and let S be differentiate on an open set containing $T(D)$. Prove that ST is differentiable on D and if $P \in D$ and $q = T(P)$ then $d(ST)|_p = dS|_q \, dT|_p$.

18. (a) Let T be of class C' on an open set D in n space, taking values in n space. Suppose that $J(P) \ne r$ for all $P \in D$. Prove that $T(D)$ is an open set.

Or

(b) Prove that the local inverses are themselves differentiable transformations and find a formula for their differentials.

19. (a) If γ is a smooth curve whose domain is the interval $[a, b]$, prove that γ is rectifiable and also show that $L(\gamma) = \int_a^b |\gamma'(t)| dt$.

Or

(b) Let F be an additive set function, defined on Θ and a.c. suppose also that F is differentiable everywhere, and uniformly differentiable on compact sets, with the derivative a point function f. Prove that f is continuous everywhere and $F(s) = \iint\limits_s f$ holds for every rectangle S.

20. (a) Let D be a closed convex region in the plane and let $w = A(x, y)dx + B(x, y)dy$ with A and B of class C' in D. Prove that

$$\iint_{\partial D} Adx + Bdy = \iint_{D} dw = \iint_{D} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dxdy.$$

Or

(b) If α is a k form and β any differential form, prove that $d(\alpha\beta) = (d\alpha)\beta + (-1)^k \alpha(d\beta)$.